★ Handle decompositions and Kirby diagrams: I

★Q: Can we draw 4-manifolds?

diagrams
Reidemeister like moves

★Ref:

Lectures on Morse theory and handlebodies
– John. Etnyre

★Outline:

(1) Handle decompositions n-manifolds.

(2) Kirby diagrams.

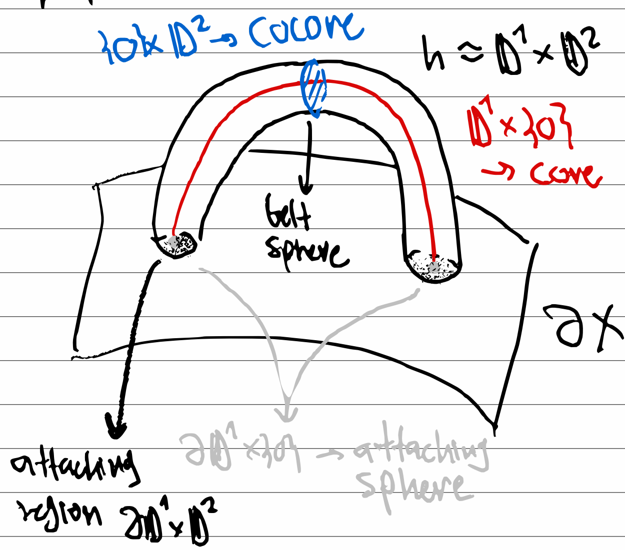
(3) Kirby moves.

(1) Handle decompositions: X n-dim. manifold w/ boundary.

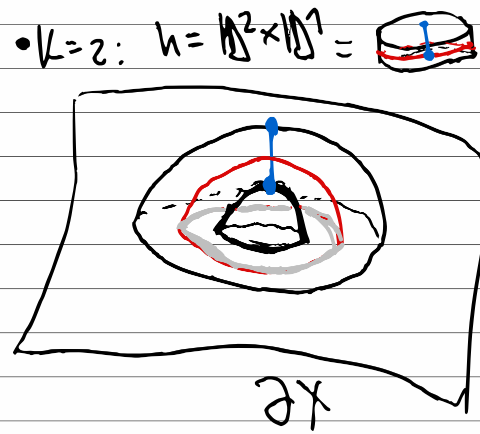
★Def: for $k=0,\dots,n$ an n-dim. k-handle h is a copy of $\mathbb{D}^k \times \mathbb{D}^{n-k}$ attached to ∂X along $\partial \mathbb{D}^k \times \mathbb{D}^{n-k}$ by an embedding $\varphi: \partial \mathbb{D}^k \times \mathbb{D}^{n-k} \to \partial X$.

• Example: $n=3$

• $k=1$:



• $k=2$: $h = \mathbb{D}^2 \times \mathbb{D}^1 =$



- Now we study how isotopying the attaching map φ of a handle changes the resulting diffeomorphism class of the manifold $X \cup_\varphi h$.

Theorem (Tubular Neighborhood Theorem):
Let $X \subset M$ smooth submanifold. Then there exists a diffeomorphism from an open neighborhood of X in $N(X,M)$ onto a open neighborhood of X in M.

We denote by $\varphi_0 := \varphi|_{\partial \mathbb{D}^k \times \{0\}} : \partial \mathbb{D}^k \times \{0\} \to \partial X$
By the TNT $\varphi : \partial \mathbb{D}^k \times \mathbb{D}^{n-k} \to \partial X$ can be constructed from an embedding $\varphi_0 : \partial \mathbb{D}^k \times \{0\} \to \partial X$ together with an identification $f : \nu \varphi_0(\mathbb{S}^{k-1}) \to \mathbb{S}^{k-1} \times \mathbb{R}^{n-k}$ and this data determines φ up to isotopy.

Since φ isotopic to φ' implies $M \cup_\varphi h$ diffeom. to $M \cup_{\varphi'} h$, the diffeomorphism type of $M \cup_\varphi h$ is specified by:

1. an embedding $\varphi_0 : \mathbb{S}^{k-1} \to \partial X$ (knot in ∂X) w/. trivial $\nu \varphi_0(\mathbb{S}^{k-1})$

2. a framing f of $\varphi_0(\mathbb{S}^{k-1})$, i.e. $f : \nu \varphi_0(\mathbb{S}^{k-1}) \xrightarrow{\simeq} \mathbb{S}^{k-1} \times \mathbb{R}^{n-k}$.

$$\begin{tikzcd} \nu \varphi_0(\mathbb{S}^{k-1}) \arrow[r, "\simeq"] \arrow[dr] & \mathbb{S}^{k-1} \times \mathbb{R}^{n-k} \arrow[d, "pr_1"] \\ & \mathbb{S}^{k-1} \end{tikzcd}$$

↳ We see the framings as sections of the $GL(n-k)$-principal bundle $V_{n-k}(\nu \mathbb{S}^{k-1})$ associated to the vector bundle $\nu \mathbb{S}^{k-1} \to \mathbb{S}^{k-1}$ and thus the obstraction to deform f_1 into f_2 is an element of $H^{k-1}(\mathbb{S}^{k-1}, \pi_{k-1}(GL(n-k))) \simeq \pi_{k-1}(GL(n-k))$ $\simeq \pi_{k-1}(O(n-k))$.
Moreover, if X is oriented, the structure group or $V_{n-k}(\nu \mathbb{S}^{k-1})$ is $GL_+(n-k) \simeq SO(n-k)$.

$$n=4 \quad : \quad \pi_{k-1}(SO(4-k)) = \begin{cases} \pi_0(SO(3)) = 0 & , \; k=1 \\ \pi_1(SO(2)) = \mathbb{Z} & , \; k=2 \\ \pi_2(SO(1)) = 0 & , \; k=3 \end{cases}$$

- For attaching 4-handles there is a unique way since any diffeomorphism of S^3 is isotopic to Id or a reflexion.

*Def: X^4 compact, connected, oriented s.t. $\partial X = \partial_+ X \sqcup \overline{\partial_- X}$.

A handle decomposition of X (rel. to $\partial_- X$) is an identification of X with a manifold obtained from $I \times \partial_- X$ by attaching handles.

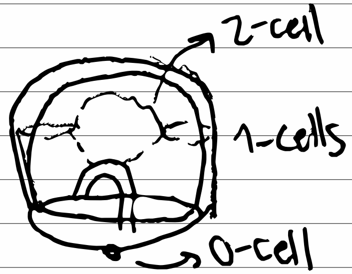
- By Morse theory we can always find Morse functions $f: X \to [0,4]$ with a lot of nice properties (notes J. Etnyre). In this case f can be chosen to satisfy

$$\begin{cases} f^{-1}(0) = \partial_- X, \; f^{-1}(4) = \partial_+ X, \text{ no critical points on } \partial X, \\ \text{self indexing, only one 0-handle (none) if } \partial_- X = \emptyset \; (\neq \emptyset) \\ \qquad \downarrow \qquad\qquad \text{'' \quad '' \quad (4) \quad '' \quad ''} \qquad (\partial_+ X = \emptyset \; (\neq \emptyset)) \\ \text{Ind } x = k \Rightarrow f(x) = k + \text{handles can be attached in any order.} \end{cases}$$

- This function f induces a CW decomposition of X with one k-cell for each critical point of index k.

$n=2$

$S^1 \times S^1$



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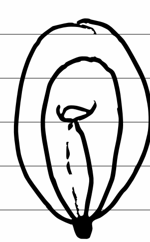
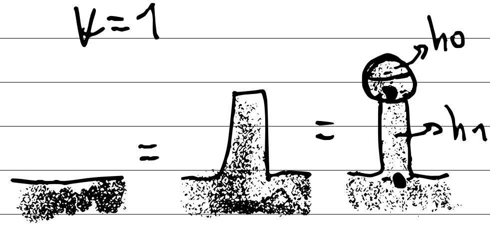


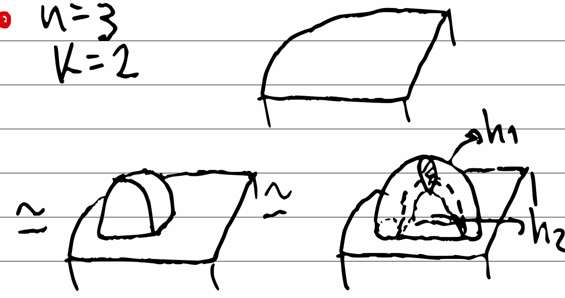
*** Prop (handle cancellation):** A $(k-1)$-handle h_{k-1} and a k-handle h_k can be cancelled if the attaching sphere of h_k intersects the belt sphere of h_{k-1} transversely in a single point. (h_k, h_{k-1} are called cancelling pair)

*** Examples:**

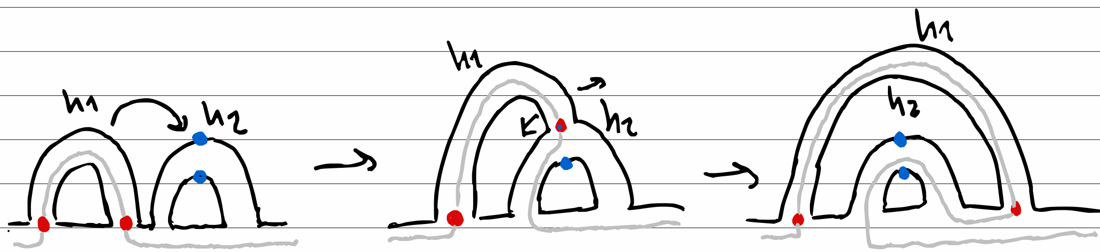
- $n=2$, $k=1$



- $n=3$, $k=2$

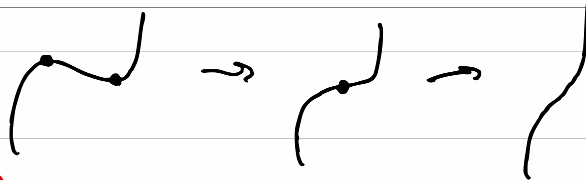


*** Def (handle slide):** Given two k-handles h_1, h_2 attached to ∂X, a handle slide of h_1 over h_2 is given by pushing the attaching sphere A of h_1 in $\partial(X \cup h_2)$ through the belt sphere B of h_2 until they intersect at one point. Then there are two possible directions for pushing A off B, one gives the original picture, the other gives the desired slide.



*** Thm (Cerf)** Given two handle decompositions of $(X, \partial_- X)$ (compact) it is possible to get from one to the other by a sequence of handle slides, creation/removal of cancelling pairs and isotopies within levels.

• **Idea:** Analyse homotopies between Morse functions.

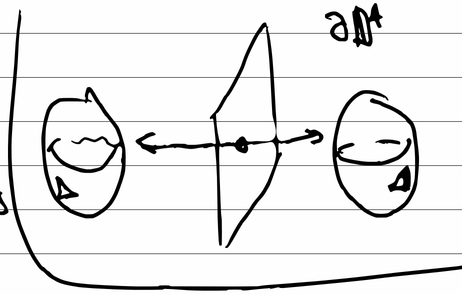


cancelling pair of critical points.

② Kirby diagrams: X compact oriented 4-man. w/. $\partial_- X = \emptyset$.

• **1-handles:** There is only one 0-handle $\mathbb{D}^4$.
The handles are glued on its boundary $\partial\mathbb{D}^4 = \mathbb{S}^3 \cong \mathbb{R}^3 \cup \infty$.
The attaching region of a 1-handle is $\partial\mathbb{D}^1 \times \mathbb{D}^3 = \mathbb{D}^3 \sqcup \mathbb{D}^3$.

There is only one framing that gives an orientable manifold.



$X^1 \to$ the union of 0- and 1-handles (thickened 1-skeleton) equals $\#_\partial n\, \mathbb{S}^1 \times \mathbb{D}^3$.

• **3- and 4-handles:** Only one framing.

↳ If $\partial_+ X = \emptyset$, as before we can assume there is only one 4-handle. By duality the union of 3- and 4-handles is diffeom. to $\#_\partial m\, \mathbb{S}^1 \times \mathbb{D}^3$ ($m = \#$ 3-handles)

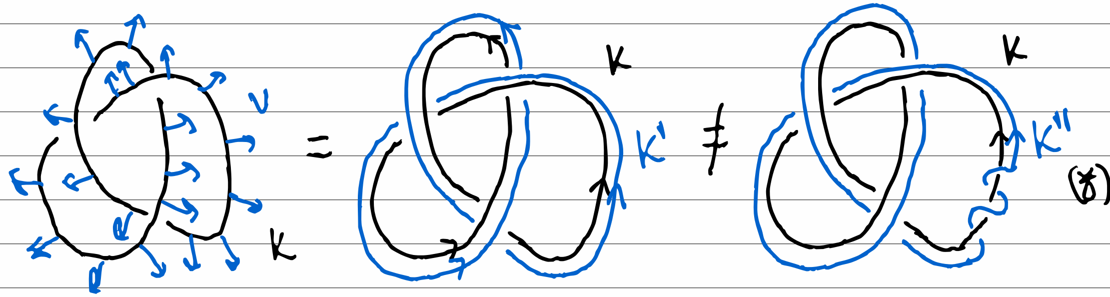
$\partial X_2 = \partial(\#_\partial m\, \mathbb{S}^1 \times \mathbb{D}^3) = \# m\, \mathbb{S}^1 \times \mathbb{S}^2 \to$ any self-diffeo. extends over X_2 (LP). Thus there is a unique 4-manifold obtained from X_2 by adding m 3-handles and with $\partial_+ X = \emptyset$.

↳ If $\partial_+ X \neq \emptyset$ and we assume that $\partial_+ X$ is connected and simply connected we do not need to deal w/ the 3- and 4-handles (Tr).

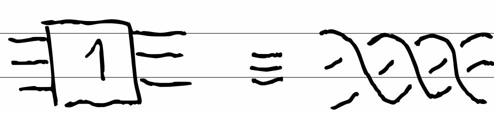
- To sum up, once we have made the diagram of the 1- and 2-handles and we have specified the number of 3-handles, there is exactly one closed 4-manifold with the given decomposition (adding one 4-handle) and exactly one simply connected 4-manifold with connected boundary with the given decomposition (and no 4-handles).

- **2-handles:** The attaching sphere of a 2-handle is a knot in $\partial X^0 = S^3$ and the framing is given by an element of $\pi_1(SO(2)) \cong \mathbb{Z}$.

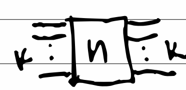
We can illustrate such an outward pointing vector field in the following way.



- **Notation:**

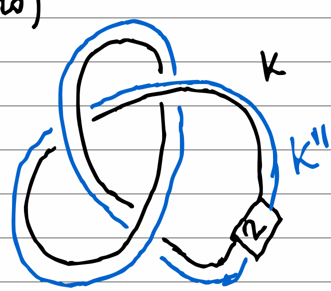


one full twist (right-handed)



n-full right-handed twists on K strands
(clock-wise)

$\Rightarrow (*) =$



- **Def:** The framing coefficient of (K, v) is $lk(K, K')$.

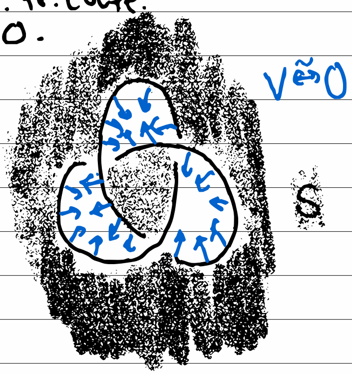
$fr(K, v) = fr(K)$

It is not obvious which framing should correspond to $0 \in \pi_1(SO(2))$. To deal with this we have the following

*Prop: • The framing coeff. of the blackboard framing on K equals the writhe $w(K)$ (the signed number of self crossings of K)

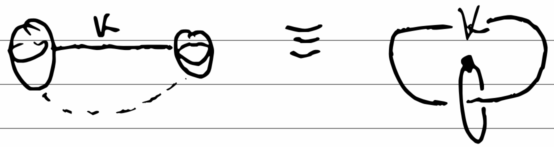
unique the framing s.t. fr. coeff. $=0$.

• The 0-framing is obtained from the outward normal to any orientable Seifert surface.



$$\text{fr. coeff}(\text{Blackb.fr}(3_1^-)) = w(3_1^-) = -3$$
$$= lk(K, K')$$

* A natural question arises when a 2-handle runs over a 1-handle. How do we deal w/ framings in the region that we do not see? Answer is new notation.

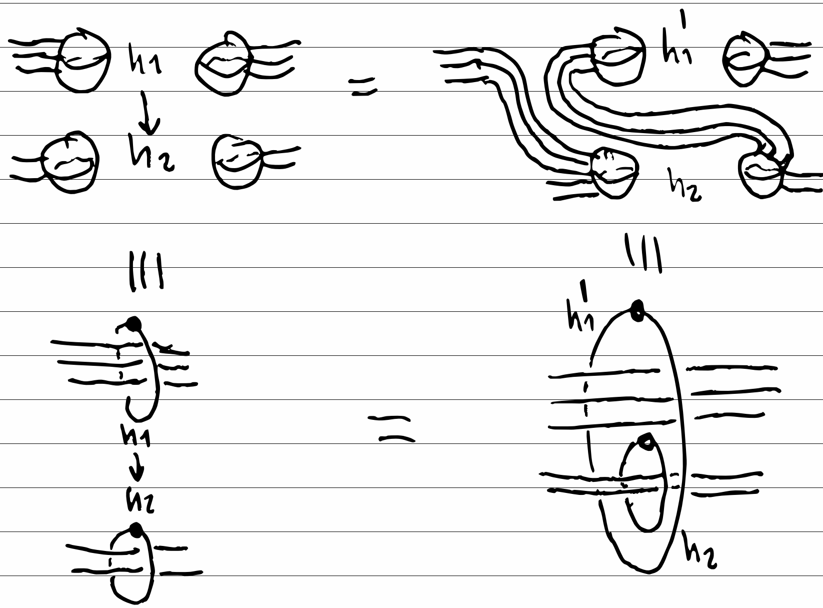


Squeezing the balls together like pancakes $\rightsquigarrow$ preserves b.b. framing.

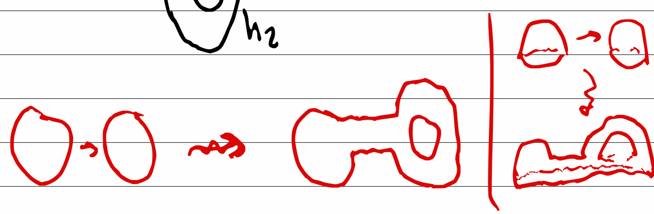
③ Kirby calculus:

The goal of this last section is to describe the complete set of moves given by Thm [Cerf] in terms of Kirby diagrams.

- 1-handle slides:



- 2-handle slides:

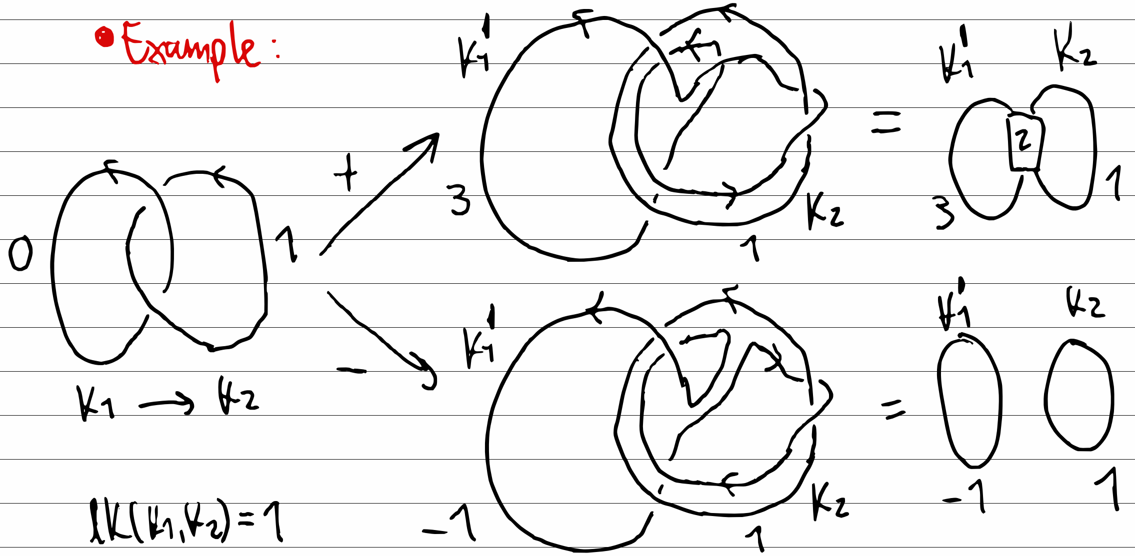


Sliding K_1 over K_2 consists of forming a copy K_2' of K_2 parallel to K_2 and that wraps around K_2 $n_2 = fr(K_2)$ times, and then replacing K_1 by the band-sum K_1' of K_1 and K_2'. We orient K_1 and K_2 arbitrarily and we endow K_1' with the orientation induced by K_1. If this orientation agrees on K_2' with the orientation of K_2 we call the move $K_1 \to K_2$ a handle addition. If it does not agree, we call it a handle substraction.

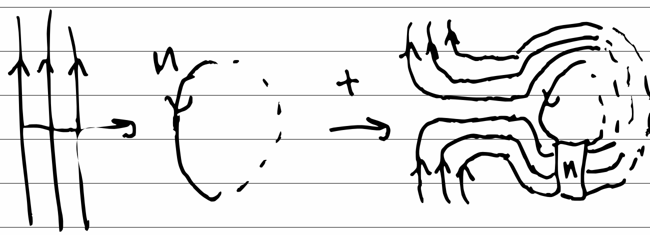
The framing coeff. of K_1' is given by the formula:

$$fr(K_1') = fr(K_1) + fr(K_2) \pm 2\,lk(K_1, K_2)$$

- **Example:**



- We can generalize this move in order to move several 2-handles over another 2-handle as follows:

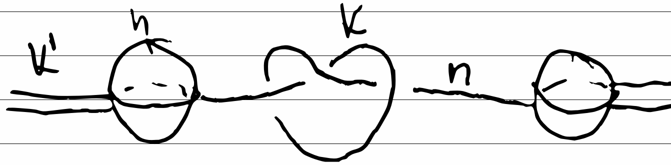


The framing depends on how the knots on the left wrap around w/ the knot on the right.

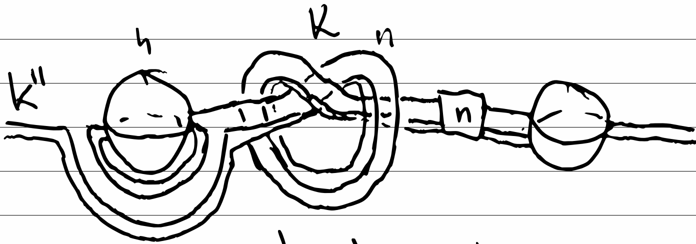
- **1/2-handle cancellation:**

- If there are no more 2-handles running over h.

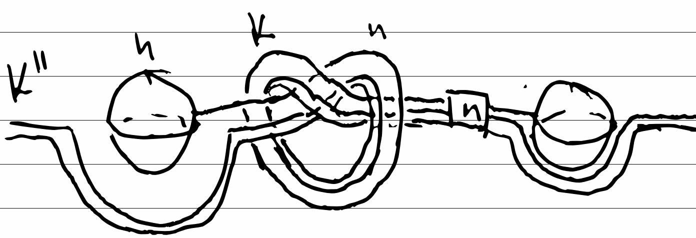
- Otherwise we slide all the other 2-handles simultaneously off h as follows.



$\downarrow K' \to K$

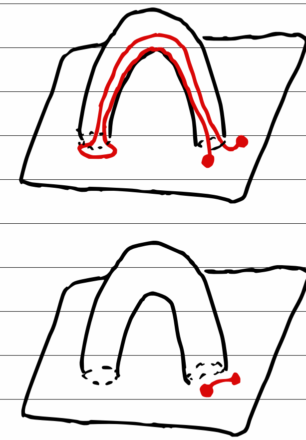


$\downarrow$ isotopy (*)

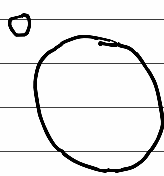


$\xrightarrow{\text{cancel } (K,h)} K''$

(*) analogy in dim 3



- 2/3-handle cancellation:



$\cup$ 3-handle $\xrightarrow{\text{cancel}} \phi$

*Prop:

- $\partial_+ X = \phi$: A 3-handle can be cancelled (after sliding 2- and 3-handles) iff we can slide 2-handles to obtain a 0-framed unknot isolated from the rest of the diagram

- $\partial_+ X$ connected, X simply connected: Erasing the 0-framed unknot and the 3-handle preserves the diffeo. type of X.